STAT 461/561 Term II, 2014-2015

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The marks of undergraduate students will be out of 40.

For Graduate students, it will be out of 45.

1. Let $(X_i, Y_i), i = 1, 2, \ldots, n$ be a set of iid bivariate observations with their joint probably density function given by

$$f(x, y; \theta_1, \theta_2) = \frac{1}{\theta_1 \theta_2} \exp(-\frac{x}{\theta_1} - \frac{y}{\theta_2}).$$

Consider the test problem for $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 > \theta_2$.

Let $\bar{X}_n$ and $\bar{Y}_n$ be sample means and define $T_n = \log\{\bar{X}_n\} - \log\{\bar{Y}_n\}$.

(a) [6] Illustrate that T_n has the desired properties for the purpose of statistical significance test.

(b) [4] Suppose the observed value of $T_n = t_0$. What is the p-value of the test based on T_n in a probability expression? Give key distributional information on how it may be computed.

Solution (a) Note that $\bar{X}_n$ and $\bar{Y}_n$ are consistent estimators of θ_1 and θ_2, respectively. If H_0 is true, we should have $\bar{X}_n/\bar{Y}_n \approx 1$ and therefore $T_n = \log(\bar{X}_n/\bar{Y}_n) \approx 0$. When H_1 is true, we would have $T_n = \log(\bar{X}_n/\bar{Y}_n) \approx \log(\theta_2/\theta_1)$ which becomes larger when H_1 is further from H_0.

By noting that X_1/θ_1 has standard exponential distribution, we would see that the distribution of $T_n = \log\{(\bar{X}_n/\theta)/(\bar{Y}_n/\theta)\}$ does not depend on the value of $\theta_1 = \theta_2 = \theta$ under H_0. Or it is a pivotal.

These are two desired properties for a test statistic.

We may add that the distribution of $\bar{X}_n/\bar{Y}_n$ is F, which is convenient for numerical computation.

(b) The p-value is defined as $P(T_n \geq t_0) = P(\bar{X}_n/\bar{Y}_n \geq \exp(t_0))$ and it can be computed according to F-distribution. Numerical integration is certainly also an option.

2. Suppose the one-parameter distribution family $\{f(x;\theta) : \theta \in \Theta\}$ has monotonic likelihood ratio in statistic $T(x)$. To make you feel right, x inside $T(x)$ may stand for the vector made of n iid observations.

   To avoid tedious technicalities, assume $T(X)$ has a continuous distribution and Θ is an interval, in addition for any $\theta_2 > \theta_1$,

$$\frac{f(x;\theta_2)}{f(x;\theta_1)}$$

   is a **strictly** increasing function of $T(x)$.

   (a) [4] Show that the most powerful test of size α for $H_0 : \theta = 1$ against $H_1 : \theta = 3$ is given by

$$\phi(T) = \begin{cases} 1 & \text{when } T > k \\ 0 & \text{otherwise} \end{cases}$$

   such that $P(T > k; \theta = 1) = \alpha$.

   Remark: you may prove it based on Neyman-Pearson Lemma or from basic. Do not simply claim the result based on a theorem given in class.

   Proof: By Neyman–Pearson Lemma, the MP test is given by

$$\phi(x) = \begin{cases} 1 & f(x;\theta = 3)/f(x;\theta = 1) > k^* \\ c & f(x;\theta = 3)/f(x;\theta = 1) = k^* \\ 0 & \text{otherwise} \end{cases}$$

   for some c and k^* such that $E[\phi(x); \theta = 1] = \alpha$.

Since $f(x;\theta = 3)/f(x;\theta = 1)$ is a **strictly** increasing function of T, there exists a k such that

$$\{x : f(x;\theta = 3)/f(x;\theta = 1) > k^*\} = \{T > k\}.$$

Since T is a continuous random variable, $P(f_3/f_1 = k^*) = P(T = k) = 0$. Hence, the MP test is equivalent to

$$\phi(x) = \begin{cases} 1 & T > k \\ 0 & \text{otherwise} \end{cases}$$

(b) [4] Give an example of monotone likelihood ratio family with continuous T that fits well with this problem. (i) identify your T and its distribution; (ii) prove the monotonicity.

Solution: Let the PDF of $x = (x_1, x_2, \ldots, x_n)$ be

$$f(x;\theta) = \theta^{-n} \exp\{-\theta^{-1} T(x)\}.$$

where $T(x) = \sum_{i=1}^{n} x_i$.

It is easy to show that this distribution family has all the properties described in (a).

(c) [4] Show that the test given in (a) is uniformly most powerful for $H_0 : \theta = 1$ against $H_1 : \theta > 1$.

Proof: The same proof given in (a) would find $\phi(T)$ is also MP of size α for $H_0 : \theta = 1$ vs $H_1 : \theta = \theta_1$ for any $\theta_1 > 1$. Hence, it is UMP against $H_1 : \theta > 1$.

3. Let $X_1, \ldots, X_n$ be a random sample from a $N(0, \sigma^2 = \theta)$ distribution, where $0 < \theta < \infty$.

(a)[4] Show that the likelihood ratio test of $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$ can be based upon the statistic $W = \sum_{i=1}^n X_i^2$.

Proof: The log-likelihood function of θ is

$$\ell_n(\theta) = -\frac{n}{2} \log \theta - \frac{W}{\theta} + \texttt{constant}.$$

Under H_1, the MLE of θ is given by $\hat{\theta} = W/n$.

Therefore, the LRT statistic is given by

$$R_n = 2\,\ell_n(W/n) - 2\,\ell_n(1) = W - n\,\log W + \texttt{constant}.$$

Since R_n depends on data only through W, the rejection region based on $R_n > c$ for any c is also a region completely determined by W. Namely, the test can be based on W.

(b)[4] State the null distribution of W.

Answer: It has chiquare distribution with n degrees of freedom under the null model.

(c)[4] Give an explicit rejection rule based on $\phi(W)$ for a size α test. Specifying the constants in $\phi(W)$ through α and some known distributions.

Solution: It is easy to see that R_n is decreasing in W when $W \in (0, n)$ and increasing in W when $W > n$. Therefore, the LRT rejects H_0 when $\{R_n > k\}$ is equivalent to reject when $\{W < k_1\}$ or when $\{W > k_2\}$ for some k_1 and k_2 such that $k_1 < k_2$ and $k_1 - n\,\log k_1 = k_2 - n\,\log k_2$.

The numerical computation can be done through the known distribution of W.

4. Let $X_1, \ldots, X_n$ be an iid sample from exponential distribution with mean parameter θ.

(a)[4] Construct the likelihood ratio test of approximate size α for $H_0 : \theta = 2$ against $H_1 : \theta \neq 2$ based on Wilks Theorem. Namely, work out the expression of R_n and specify the rejection region.

Solution: The log-likelihood function of θ is

$$\ell_n(\theta) = -n \log \theta - \frac{n \bar{X}_n}{\theta}.$$

The MLE of θ under H_1 is $\hat{\theta} = \bar{X}_n$, and hence the LRT statistic is given by

$$R_n = 2\,\ell_n(\bar{X}_n) - 2\,\ell_n(2) = n\,(\bar{X}_n - 2) - 2n \log(\bar{X}_n/2).$$

According to Wilks's theorem, R_n is asymptotically χ_1^2 distributed under H_0. Hence the LRT of approximate size α is

$$\phi(x) = \begin{cases} 1 & R_n > \chi_1^2(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

where $\chi_1^2(\alpha)$ is the upper 100α-th quantile of χ_1^2 distribution.

(b)[6] Derive the corresponding statistics for score test and Wald-test.

Solution: The score function is

$$S_n(\theta) = \ell_n'(\theta) = \frac{n\,(\bar{X}_n - \theta)}{\theta^2}$$

and the Fisher information under H_0 is

$$I(\theta) = E\left\{\frac{2X}{\theta^3} - \frac{1}{\theta^2}\right\} = \frac{1}{\theta^2}.$$

The score test statistics for $H_0 : \theta = 2$ is hence given by

$$\frac{\{S_n(2)\}^2}{nI(2)} = \frac{n(\bar{X}_n - 2)^2}{4}.$$

Similarly, the Wald test statistics is given by

$$[\sqrt{n\,I(2)}(\bar{X}-2)]^2 = \frac{n(\bar{X}_n-2)^2}{4}$$

which coincides with the score test statistic.

(c)[4] Assume the knowledge of $\bar{X}-\theta = O_p(n^{-1/2})$. Directly verify that the likelihood ratio statistic has chisquare limiting distribution with one degree of freedom under the null hypothesis (i.e. Wilks Theorem).

Proof: All discussion below is under $H_0 : \theta = 2$.

We approximate R_n by its Taylor series about $\bar{X}_n$ up to second order at 2:

$$\begin{aligned} R_n &= n\,(\bar{X}_n-2) - 2n\,\log\left(\frac{\bar{X}_n-2}{2}+1\right) \\ &= n\,(\bar{X}_n-2) - 2n\left[\frac{\bar{X}_n-2}{2} - \frac{(\bar{X}_n-2)^2}{8} + o_p((\bar{X}_n-2)^2)\right] \\ &= \frac{n\,(\bar{X}_n-2)^2}{4} + o_p(n(\bar{X}_n-2)^2). \end{aligned}$$

Since $\bar{X}_n - 2 = O_p(n^{-1/2})$, we see that $o_p(n(\bar{X}_n-2)^2) = o_p(1)$.

Note that $\sqrt{n}\,(\bar{X}_n-2) \to N(0,2)$ by the central limit theorem. We conclude that R_n is asymptotically χ^2_1 distributed according to the Slutsky's theorem.